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CERTAIN UNDEFINED ELEMENTS AND TACIT ASSUMPTIONS IN THE FIRST BOOK OF EUCLID'S ELEMENTS.*

BY HARRISON E. WEBB.

For twenty-one centuries the spirit of Euclid has reigned supreme over elementary geometry. For the greater part of this time the terms "Euclid's Elements" and "Geometry" have been all but synonymous. In no other science save theology has a single book held undisputed sway for unnumbered generations. Educational reformers have often pointed to this universal acceptance of Euclidean authority as an evidence of dead conservatism on the part of teachers. It is due rather to recognition of the unparalleled skill of the great Elementarist, whose text still stands as the best among the thousands which have been written on the subject, and whose methods are so simple and direct as to be well within the abilities of a beginner.

The beginner in the study of geometry, nevertheless, encounters certain difficulties in detail which must be considered in teaching the subject. These tend to minimize interest on the part of the student and to make the task of the teacher doubly hard. In consideration of this fact the writer wishes to submit the following pedagogical principle in mathematics: that *where any considerable number of normally intelligent students, under*

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* Presidential address before the Association of Mathematics Teachers of New Jersey, read at a joint session with the Association of Mathematics Teachers of the Middle States and Maryland.

varying conditions, find serious difficulty in the mastery of a certain detail, the error is less likely to be in the teaching than in the development of the subject itself. This principle, if sound, would indicate that Euclid made mistakes, or, at least, that there are errors in the Euclidean text as we have it. We may as well admit that such is the case, though they are few in number. In the effort to correct these we can scarcely do better than to follow Veblen's dictum: *to endeavor to rewrite the beginning of geometry as Euclid might have written it.* But this is a very high standard for the experimenter to set for himself, and he is much more likely to fail than to succeed.

So far as we know, Euclid nowhere states explicitly his purposes in formulating a science of space relations. A perusal of the theorems of Book I, however, seems to bring into clear relief the following aims on the part of the author:

1. To show that there is only one geometry.
2. To set a standard of logical procedure.
3. To set forth the fact that geometry begins with a limited number of preliminary statements, so simple in character as to be undeniable—"self-evident," as they are sometimes called.
4. To reduce the number of these preliminary statements to a minimum.
5. To separate the geometry of space in two dimensions from that of space in three dimensions.
6. To show from certain well-known configurations the existence of other configurations.

Editors of the Euclidean text for the use of schools have adhered to these aims. Authors of geometrics based upon Euclid have, as a rule, departed definitely from one or another of them. Probably the best known of the substitutes for Euclid's Elements is the geometry of Legendre, whose work has formed the basis of most of the American texts of the past century. Euclid, in pursuit of the sixth of the aims just mentioned, endeavors to account for every line and point demanded in a demonstration by a previous postulate or construction. Legendre, in establishing a new sequence, has added to the postulates two, to the effect that a line has a midpoint, and that an angle has a midray, both of which configurations he later constructs by processes which depend for their validity upon certain theorems which employ the construction themselves. Legendre's American successors have as a rule gone further and introduced into their demonstrations any configuration which pleased them, thus departing entirely from Euclid's rule in the matter. This seems to be a serious pedagogical blunder, as well as a logical error which students are not slow to notice. To tell a student offhand that this or that line or point exists, without having proved its existence and shown how to establish it, is bad teaching and worse geometry. Recent texts in America have shown a tendency to return to Euclid in this particular.

The question before us is: Wherein, if at all, does Euclid himself fail in respect to the six aims referred to.

With regard to the first, it may as well be understood that he failed utterly. It is now known that there are many geometries based upon as many sets of assumptions. One of these, for historical reasons, is called Euclidean. The others have various names, which need not concern us here.

As to the second aim, Euclid's logic is unquestionably superior to that of any of his successors, up to the present period, which has been marked by a most careful examination into the processes of symbolic logic; the names of Hilbert, Pieri, Cantor, Dedekind and Russell come at once to mind, and in our own country those of Moore, Veblen, and Huntington. It is safe to say, however, that these mathematicians have set for themselves a general standard hardly higher than that of Euclid, but, instead, in the field of geometry, have been interested chiefly in correcting his mistakes.

Regarding the third aim: Euclid's axioms and postulates are all self-evident, except the one numbered twelve in the list given. About this one, the famous parallel postulate, an immense literature has been written. It looks as if it *ought* to be demonstrable—whence many sleepless hours and wearied intellects, and an abundant flood of ink, from the days of Proclus to the dreams of the latest neophyte. Its independence was fully established by Hilbert.

The real rub comes, however, with the fourth and fifth aims just mentioned. The latter is in a measure artificial, and many modern writers abandon it entirely. The present basis for preserving the distinction between plane and solid geometry is that of convenience in arrangement of time units in teaching the subject. This has led to the rather ridiculous result that in many schools a highly theoretical plane geometry is required of all students, while an extremely practical solid geometry is elected by only a few.

As to a minimum list of assumptions, the idea came to prevail in the study of geometry that *nothing should be assumed which could be proved.* Whether Euclid had this idea we do not know. Certainly many of his theorems are "self-evident" enough; but he proves them with very great care. On the other hand there are many concepts and ideas, seemingly deserving of explicit statement, which he mentions not at all. I believe that this fact constitutes the great stumbling block to beginners in the study of Euclid. The *pons asinorum,* for example, is really of faulty construction, and the much-maligned beast, like his predecessor who bore Balaam, often shows keener powers of perception than his master.

A set of assumptions, to pass muster nowadays, must meet three requirements, designated as follows:

A. *Consistency,* meaning that they do not lead to contradiction.

B. *Independence,* meaning that no one of them can be deduced from the others.

C. *Categoricalness,* meaning (roughly) that they characterize the science for which they are devised, and no others.

Euclid's axioms and postulates do not meet these tests. A set of assumptions which will do so constitutes the absolute beginning of the subject. A number of such sets have been written, though the proofs that they meet the foregoing conditions are not complete. The set which is probably the best known in this country is that of Veblen, which is derived from projective geometry. To this set we will return presently.

Euclid's Elements have been edited, of course, many times. The best edition, one which should be available to every teacher of geometry, is that of Sir Thomas Heath (Cambridge, 1908). The references given below are to Todhunter's edition, long accepted as a standard. It can hardly be said of Todhunter, as may freely be asserted of many authors of geometries, that he has added to the number of logical errors which occur in Euclid.

Considering Todhunter's text, then, we may note first as to the *definitions:*

Numbers 1 and 2 do not define at all. The idea of length is much more complex than that of line. Number 4 defines "evenly" but not "straight line." Number 7 combines a definition with an assumption, neither of which is clearly stated. Numbers 8 and 9 define "inclination," and "different directions," but not "plane angle." Number 10 presupposes the existence of two lines under the given conditions, and also presupposes the existence of equal angles, a fact which Euclid is at great pains to prove in Book I, Prop. 23. Number 15 presupposes the existence of equal lines through one point. Of the other definitions it may well be that, if they were distributed at various points in the text, they would escape the error of involving "presuppositions." It is quite conceivable that Euclid so intended them to be used. Of the terms referred to, however, attempts in elementary texts at rigorous definition have only made a bad matter worse, as a rule.

As to the postulates:

The distinction between "Postulates" and "Axioms" is due not to Euclid, but to later editing. It is, of course, a fanciful one. Following the familiar usage, however, Postulate 1 should be combined with Axiom 10, to read "One and only one straight line can be drawn." Postulate 2 is generally superseded, now-a-days, by a distinction in definition between "line" and "line-segment." As it stands it is a play upon the word "straight." Axioms 1 to 7 overlook entirely the differences between the uses of the word "equal" when applied to lines, angles and areas. They really presuppose an assumption to the effect that each of these three things stands in direct one-to-one correspondence with the number system of Algebra. This is true, of course, but not "self-evident," as is shown by many familiar illustrations.

Axiom 9 is "axiomatic" enough to satisfy the most fastidious, but it is employed in numerous demonstrations without the slightest effort to *prove* which of two magnitudes is the whole and which is the part.

In axioms 11 and 12 Euclid is on sound foundations, provided a process of "taking together" is defined. This he does not do, however.

If we now turn to the propositions:

Proposition 1 assumes that the two circles he describes will intersect. This seems obvious enough in this case, but so do a large number of other things which he, and we after him, are at great pains to prove; as in his own very next propositions, which the stomach of a present-day teacher of geometry revolts at, preferring to assume, without statement, that a compass will "stay put" when it is taken from the paper.

In proposition 4, however, Euclid throws caution to the winds and deliberately picks up a triangle as if it were a biscuit and deposits it upon another one. This operation has probably caused more annoyance to teachers and students than any other one thing in the beginning of geometry. I believe that every American text but one (Halstead) employs the same method of proof. One of my students a few days ago characterized it as "pure bunk."

Countless explanations are printed or offered orally. The first is to the effect that there is really an assumption of geometry which states that any figure can be moved from one place to another without altering the magnitude of its sides, its angles or its area. This assumption has high authority, but, as Russell points out, it presupposes motion as a geometric process, whereas motion itself requires to be defined in geometric terms. "Rigidity" of a geometric concept is about as conceivable as rigidity of a happy thought or of a devout hope. Pieri's list of assumptions for the foundation of geometry begins with motion. But this alters entirely the basis of the subject, and eliminates the necessity for a large number of propositions which we prove with assiduous regularity.

Proposition 5 now demands out attention. It is true that Euclid nowhere in Book I explicity states that all points are in the same plane. He presupposes it, however, in Proposition 30, and makes no reference to points outside of his fixed plane before Book XI. It is perfectly apparent, however, that the dictum (due, I think, to Pascal) of substituting the proof for the thing proved cannot be applied here. For to bring any pair of triangles from the figure into coincidence with each other it is necessary not only to move one of them, but to *turn it bottom-side up,* which involves definitions of the top-side and the bottom-side of a plane triangle, and Heaven only knows what else, including space of three dimensions.

Propositions 6 and 7 taken together really constitute the proof of the third triangle congruence (three sides respectively equal); they are open to the same difficulty as the preceding. It is universally admitted that the familiar proof of this theorem is an improvement upon the method of Euclid. In 6, moreover, as Russell points out, Euclid first employs an axiom of which he is apparently wholly unconscious, though it is most essential to his sequence, viz.: that if AB, AC, and AD meet a line not through A in three points B, C, and D, then if B lies between C and D, then the angle BAC is less than the angle BAD; and in 7 there is need for a further assumption, that if the same three lines are cut by a fifth line in B', C', and D', then if C lies between B and D, C' must lie between B' and D'. In passing it may be noted that this axiom does not apply in elliptic space.

Proposition 12 employs without proof or explicit statement the fact that a straight line has two sides, and also that a circle cuts a straight line, if at all, in two and only two points.

Proposition 16 is open to the same objection as proposition 7. This is important when it is considered that it is upon this theorem that Euclid's whole theory of parallels rests.

Proposition 23 proves the existence of an angle when a given line at a given point equal to a given angle. It makes no mention of the important fact that there are two such. It should be noted, moreover, that without this proposition the student cannot construct the conditions laid down for Proposition 4. It looks very much as if Euclid had had as much trouble with Proposition 4 as the rest of us, and finally given the proof up as a bad job.

But enough of fault finding. Let us turn to our moral lessons. How should these and similar points of criticism effect our teaching of geometry?

The first lesson to be drawn is this: that no sequence in elementary geometry, not even that of Euclid, is sacred. On the other hand, even a hasty perusal of the fundamental principles of geometry as formulated by Hilbert, Pieri or Veblen, convinces the reader that the statement of them involves a degree of abstraction far too profound for the mind of the beginner in the subject. In other words, geometry, like any other science, has its psychological as well as its logical origins, and these should be employed fully in the first presentation of the subject, instead of plunging the student unprepared into a sea of doubtful logic, as has been done in the past.

It is easy, however, to err in the other extreme of throwing logic entirely to the winds, and substituting empirical tests for every demonstration which seems a little difficult or unusual.

The better plan is to point out explicitly certain elements which are left undefined, and certain propositions which remain unproved, establishing their reasonableness by experiment, and to follow these up with accurate definitions of other terms and logical demonstrations of other theorems. In the choice of these undefined elements and unproved propositions, it may be well to follow Euclid rather closely. The "assumptions of order," should be assumed tacitly at the outset; they can be mentioned informally when needed. Clearly enough Euclid really employs many undefined elements. We have seen that many of his definitions are faulty. Others are too abstract to be grasped by the beginner. Certain of these geometric elements we should specify by name so as to employ them in defining other configurations:

Point
Straight line
Curved line
Plane
Curved surface
Equality of lines
Equality of angles
Equality of areas

Confusion only is the result when the three equalities mentioned are "defined" by superposition. It is by no means difficult for the beginner to grasp the contrary idea, that superposition is defined by equalities. This involves the omission of propositions 1, 2 and 3 from the Euclidian sequence.

Hilbert, as is well known, states Euclid I, 4, in part at least, as an assumption; the other two congruence theorems for plane triangles are easily derived from this, as Euclid shows, by indirect method. In general, it is probably the best procedure for beginners to regard all three as assumptions, and along with them Euclid's two tacit assumptions regarding circles, referred to above. This leaves untouched, however, the difficulty which the student encounters with Euclid I, 5 (the *pons asinorum*), and with the introduction to the study of parallel lines.

Veblen's assumptions are fewer in number than those of Hilbert, and their significance is more easily grasped. As they constitute a satisfactory foundation for metrical geometry, it is desirable to note any necessary points of departure from them in elementary teaching.

Assumptions I to VI, in order to meet as closely as possible the criteria of independence, consistency and categoricalness, and for accuracy in definition, are so phrased that a beginner would hardly grasp their significance. For them may be substituted, in introducing the beginning to the subject, the familiar criteria for locating points on a straight line.

Assumption VII defines three-dimensional geometry, and Assumption VIII limits space to three dimensions at most. These belong to solid geometry. For plane geometry, VII would read:

All points are in one plane.

Considering now assumption IX, which, as Legendre shows, is the equivalent of Euclid's "parallel postulate." Hilbert makes this assumption read: ". . . there is *one and only* one line coplanar with *a* and not meeting *a*." Let us accept this form in spite of Euclid I: 27. Suppose now that "parallel lines" and "parallelogram" are defined in the usual fashion, and that there is added this assumption:

Opposite sides of a parallelogram are equal.

If now to equal and parallel line-segments we apply assumptions X to XIII, we have the foundations of a geometry independent of "rigid motion," and involving a large number of familiar propositions, which are demonstrable with Euclidean exactitude, but by methods which are simpler than those usually employed. Among these are:

Two equal and parallel lines determine a parallelogram.

To double a straight line.

To bisect a straight line.

The midline of a triangle is parallel to the base and equal to one half of it.

To divide a line into any number of equal parts.

The diagonals of a parallelogram bisect each other.

In this development the familiar "subtraction, multiplication, and division axioms," as applied to parallel straight lines, may be stated as assumptions, though they are in reality theorems.

The definition of "angle" now demands our attention. Euclid's definition, as has been said, is unsatisfactory. But if the familiar definition as "the figure formed by two rays having a common origin" is accepted, with the usage of "straight angle" (an excellent idea with an unhappy name), then Euclid I: 13, 14, 15, and certain of the familiar definitions of various kinds of angles follow at once. The sum and the difference of two angles are also definable, when the angles have a common vertex.

Assumption XIV in Veblen's set serves, as he has shown, to establish equality of angles generally, and to replace Hilbert's axiom regarding congruent triangles. As a partial substitute for this assumption, let us state the following:

Corresponding (exterior-interior) angles of parallel lines are equal.

This assumption opens the door for the usual sequence regarding angles and parallel lines, including the proof for the sum of the angles of a triangle.

We may call the geometry thus obtained the GEOMETRY OF TRANSLATION. (It also involves in some measure the idea of point-reflexion, but this is hardly essential to the sequence.)

The geometry of translation does not permit of the equality of lines which are not parallel. This concept is introduced by Euclid's definition of the circle. One of Euclid's tacit assumptions may now be stated explicitly, to the effect that:

A straight line may have not more than two points in common with a circle. And this should be followed by Veblen's assumption XV.

Let us now add arbitrarily this assumption:

When two circles intersect, the line-segments joining their points of intersection to any given point on the line of centers are equal. These line-segments are called *symmetric* segments, and their corresponding extremities *symmetric* points.

It will be noticed at once that this assumption states in geometric form a fact which is illustrated in paper folding. The geometry of paper folding has been developed with considerable detail. It is necessary, however, from a purely geometrical point of view, to account for the physical phenomena of paper folding in terms of formal definitions, assumptions, and propositions. *Symmetric rays* and *symmetric lines,* in particular are readily defined.

The analogous assumption regarding angles in the GEOMETRY OF SYMMETRY appears to be this:

An angle between two rays is equal to the angle between their symmetric rays. This assumption includes as a limiting case one to the effect that *symmetric rays make equal angles with the axis.*

It is now assumed that the axioms of equality extend to equal lines and equal angles as thus defined.

This geometry has been developed at some length in many texts. On the basis above given it is independent of parallelism, and is applicable to hyperbolic space, or to elliptic space when necessary restrictions are placed upon the radii of the intersection circles. Symmetry also defines the right angle, and justifies most of the familiar constructions of geometry. The equality of the base angles of an isosceles triangle is a particular case of symmetry. It cannot be proved in Euclidean fashion without some form of "folding over."

The theorem "A line perpendicular to a radius of a circle at its extremity is tangent to the circle" is also proved by symmetry, as are also the theorems "The bisectors of the angles of two intersecting lines are the locus of the centers of circles tangent to the two lines," and "The perpendicular bisector of a line joining two given points is the locus of the centers of circles passing through the two points."

The principles of symmetry, when properly formulated, also hold for spherical geometry as that subject is usually defined. It is to be noted merely that the physical phenomenon of paper folding is not applicable here.

The third of the introductory steps to geometry has to do with *rotation.* If motion is employed for illustration only, and not for demonstration, certain assumptions are needed, which can be stated most clearly by adding one to two new terms to the vocabulary of the subject.

Two points are said to be *concentric* about a third point when they lie on a circle of which the third point is the center. Two lines are said to be *concentric* about a point if they are tangent to a circle of which the point is the center. Two rays are *concentric* if their vertices are concentric and the lines of which they form a part are concentric about the same point, and if either they both do or both do not include the points of tangency.

Two segments are *concentric* if they are laid off on concentric rays and their extremities are concentric about the same point as are the rays.

Two angles are *concentric* if their sides and vertices respectively are concentric about the same point, and if either they both do or both do not contain that point.

From these definitions and the locus theorems referred to above it is easy to construct the center for two given non-parallel rays. In so doing it will be found that in general *two* locus intersections are obtained. The one should be chosen which is or is not separated from the vertex by the line through the origin of the rays, according as the rays themselves are or are not so separated.

The two assumptions of this geometry can now be stated:

Concentric segments are equal, and

Concentric angles are equal.

To these may be added the familiar proposition:

In the same circle, equal chords subtend equal central angles.

This means in terms of motion that of two such segments one can be rotated about their common center to coincidence with the other, and that of two such angles one can be rotated about their common center either to coincidence with or to a position symmetrically adjacent to that of the other.

The theorems, constructions and special instances of this GEOMETRY OF ROTATION are for the most part unfamiliar, but they are relatively simple and very interesting.

We now come to the subject of congruent triangles.

It appears that if in the triangles ABC and DEF, located at random, AB equals DE by the principles of the geometry of translation, or of that of symmetry, or of that of rotation, it is possible to construct on the line DE a triangle $A'B'C'$ whose sides and angles will be respectively equal to those of the triangle ABC. This triangle is said to be *congruent* to the triangle

ABC, the term "congruent," in this instance, having the same meaning as Euclid's "equal in all respects." If on the line *DE* there is already a triangle *DEF* such that $DF = A'C'$ and $EF = B'C'$ then the angles of triangle $A'B'C'$ will either coincide with or be symmetrical to those of triangle *DEF*; for otherwise two circles would have three distinct points in common.

It is customary in plane geometry to call symmetric triangles congruent. Euclid does this tacitly in Proposition I, 5, as has been seen. If this is done explicitly, we have proved the third congruence theorem in its usual form.

The other two congruence theorems are established in similar fashion.

The foregoing outline for an introduction to geometry is based, as is apparent, upon the substitution of eight or nine assumptions in place of Veblen's assumptions IX and XIV.

From the point of view of reducing the number of assumptions to a minimum this is a palpable surrender of principle. But it is now fully understood that many, if not most, of the assumptions of metrical geometry are in reality demonstrable propositions of projective geometry, and that the Euclidean geometry of our forefathers is really one of many "subgeometries" of projective geometry, namely, the parabolic metric geometry. To add to the list of assumptions of Euclidean geometry, then, is merely to make a new selection from the theorems of projective geometry, and the above list is suggested as an experiment in this direction.

Of the pedagogic advantages which this procedure seems to suggest the following may be noted:

1. An analysis of motion is included which is closely related to the study of kinematics.

2. The methods of construction are brought into closer relation with those which prevail in mechanical drawing.

3. The principles as laid down appear first as direct inferences from processes of drawing, the instruments being the straight edge, the ruler and some form of parallel-line machine.

4. Many demonstrations are reduced to much simpler terms. There is need at this time for closer attention to what may be called the "psychology of logic"—the nature of a definition, and the offence of unnecessary circumlocution.

5. Vector analysis has a well-defined point of contact with elementary mathematics.

In short a channel is opened for a course in inventional geometry which can later be brought into harmony with the deductive science on either a relatively or a purely formal basis. And the "axiom of parallels" thus appears in its proper perspective rather than as a miraculous interference with the processes of human logic.

For convenience of reference the following captions taken from Todhunter's Edition of Euclid and from Veblen's Projective Geometry (Vol. II) are subjoined.

EUCLID'S ELEMENTS.

Definitions.

1. A point is that which has no parts, or which has no magnitude.
2. A line is length without breadth.
3. The extremities of a line are points.
4. A straight line is that which lies evenly between its extreme points.
5. A superficies is that which has only length and breadth.
6. The extremities of a superficies are lines.
7. A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.
8. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.
9. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.
10. When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.
11. An obtuse angle is that which is greater than a right angle.
12. An acute angle is that which is less than a right angle.
13. A term or boundary is the extremity of any thing.
14. A figure is that which is enclosed by one or more boundaries.

15. A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

16. And this point is called the center of the circle.

17. A diameter of a circle is a straight line drawn through the center, and terminated both ways by the circumference.

(A radius of a circle is a straight line drawn from the center to the circumference.)

Postulates.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line:

3. And that a circle may be described from any center, at any distance from that center.

Axioms.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Propositions.

1. To describe an equilateral triangle on a given finite straight line.

2. From a given point to draw a straight line equal to a given straight line.

3. From the greater of two given straight lines to cut off a part equal to the less.

4. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

5. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

6. If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

7. On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.

8. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

9. To bisect a given rectilineal angle, that is to divide it into two equal angels.

10. To bisect a given finite straight line, that is to divide it into two equal parts.

11. To draw a straight line at right angles to a given straight line, from a given point in the same.

12. To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

13. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

14. If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

15. If two straight lines cut one another, the vertical, or opposite, angles shall be equal.

16. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

17. Any two angles of a triangle are together less than two right angles.

18. The greater side of every triangle has the greater angle opposite to it.

19. The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

20. Any two sides of a triangle are together greater than the third side.

21. If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

22. To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

23. At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

24. If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.

25. If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them, of the other.

26. If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other

sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

27. If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

28. If a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side equal to two right angles, the two straight lines shall be parallel to one another.

29. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

30. Straight lines which are parallel to the same straight line are parallel to each other.

31. To draw a straight line through a given point parallel to a given straight line.

32. If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

33. The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Veblen's Assumptions.

1. If points A, B, C are in the order $\{ABC\}$ they are distinct.

2. If points A, B, C are in the order $\{ABC\}$ they are not in the order $\{BCA\}$.

3. If points C and D $(C = D)$ are on the line AB, then A is on the line CD.

4. If A and B are two distinct points there exists a point C such that A, B and C are in the order $\{ABC\}$.

5. If three distinct points A, B and C do not lie on the same line and D and E are two points in the orders $\{BCD\}$ and (CEA), then a point F exists in the order $\{AFB\}$ and such that D, E and F lie on the same line.

6. There exist three distinct points, A, B, C, not in any of the orders $\{ABC\}$, $\{BCA\}$, $\{CAB\}$.

7. If A, B, C are three non-collinear points, there exists a point D not in the same plane with A, B and C.

8. Two planes which have one point in common have two points in common.

9. If A is any point and a any line not passing through A, there is not more than one line through A coplanar with a and not meeting a.

10. If $A = B$, then on any ray whose origin is C there exists one and only one point D such that (A, B) is congruent to (C, D).

11. If (A, B) is congruent to (C, D) and (C, D) is congruent to (E, F) then (A, B) is congruent to (E, F).

12. If (A, B) is congruent to (A', B') and (B, C) is congruent to (B', C') and $\{ABC\}$ and $\{A'B'C'\}$, then (A, C) is congruent to (A', C').

13. (A, B) is congruent to (B, A).

14. If A, B, C are three non-collinear points and D is a point in the order $\{BCD\}$, and if $A'B'C'$ are three non-collinear points and D' is a point in the order $\{B'C'D'\}$ such that the point-pairs (A, B), (B, C), (C, A), (B, D) are respectively congruent to (A', B'), (B', C'), (C', A'), (B', D') then (A, D) is congruent to (A', D').

15. If the line joining the centers of two coplanar circles meets them in pairs of points P_1Q_1 and P_2Q_2 respectively such that $\{P_1P_2Q_1\}$ and $\{P_1Q_1Q_2\}$ the circles have two points in common, one on each side of the line joining the centers.

16. If A, B, C, are three points in the order $\{ABC\}$ and B_1, B_2, B_3 are points in the order $\{ABB_1\}$, $\{AB_1B_2\}$ . . . such that (A, B) is congruent to each of the point-pairs (BB_1), (B_1B_2) . . . , then there are not more than a finite number of points B_1, B_2 . . . between A and C.

Suggested Changes and Additions for Elementary Instruction.

9. ". . . there is one and only one line. . . ."

Let "equal" be substituted for "congruent" as applied to lines and angles.

10*a*. If the rays (*a*, *b*) form an angle, and *c* is a third ray, then with *c* there are two and only two angles each equal to the angle (*a*, *b*).

11*a*, 12*a*, 13*a*, involve applications of the principles of equality to angles.

14. To be omitted.

T1. Opposite sides of a parallelogram are equal.

T2. Corresponding angles of parallel lines are equal.

S1. "Symmetric" segments are equal.

S2. "Symmetric" angles are equal.

R1. "Concentric" segments are equal.

R2. "Concentric" angles are equal.

R3. In the same circle equal chords subtend equal central angles.

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